J. Fluid Mech. (1980), *vol.* 101, *part* 1, *pp.* 65–78
Printed in Great Britain

The rate of magnetic field penetration through a Bénard convection layer

By E. M. DROBYSHEVSKI, E. N. KOLESNIKOVA AND V. S. YUFEREV

A. F. Ioffe Physical-Technical Institute, U.S.S.R. Academy of Sciences, Leningrad K-21

(Received 2 October 1978 and in revised form 7 January 1980)

Non-stationary MHD interaction of a horizontal magnetic field with a three-dimensional cellular convection is studied by means of computational methods and methods of mean field electrodynamics.

For a given magnetic field drop across the convective layer, the rate of magnetic flux penetration through this layer is characterized by two integral coefficients: the first one describing the topological pumping effect arises from the antisymmetric part of the α-effect, while the second coefficient accounts for the enhancement of the effective diffusion due to the convective motions. In the magnetic-Reynolds-number range studied ($-5 \leqslant R_m \leqslant 5$) these coefficients are found to be, correspondingly, odd and even functions of R_m only. The net magnetic flux escape rate into vacuum decreases at $R_m > 2{\cdot}2$ when compared with a case of a layer without cellular motions. Here the topological pumping prevails not only over the convective enhancement of diffusion but begins to suppress even the background diffusion action.

Thus, the asymmetry in the transport properties of cellular motion is again demonstrated, and their difference from those of random turbulence is identified.

1. Introduction

The interaction of a magnetic field with three-dimensional cellular (Bénard) convection of a conducting fluid was shown by Drobyshevski & Yuferev (1974; hereinafter referred to as I) to be essentially different from the interaction with irregular turbulence. Physically, this difference is due to the topological properties of such convection, when flows of the liquid in one direction form a continuous mesh which traps in its cells flows of opposite direction isolated from one another. As a result, the transfer of continuous magnetic tubes of force occurs differently from that of a scalar admixture. The flows making up a continuous mesh are capable of trapping and transporting the tube of force as a whole. Opposite discrete flows cannot carry back the tube of force as a whole, and instead tear off from it closed loops which do not carry net magnetic flux. Thus, a three-dimensional convection layer operates as a pump creating and maintaining a magnetic-field drop between the base and the surface of the layer.

Generally speaking, it is clear that any asymmetry in the flow or parameters of a conducting liquid within a layer should result in some pumping or enhanced escape of magnetic field despite the absence of net mass motion (I, p. 40). Similar effects

0022-1120/80/4571-2290 $02.00

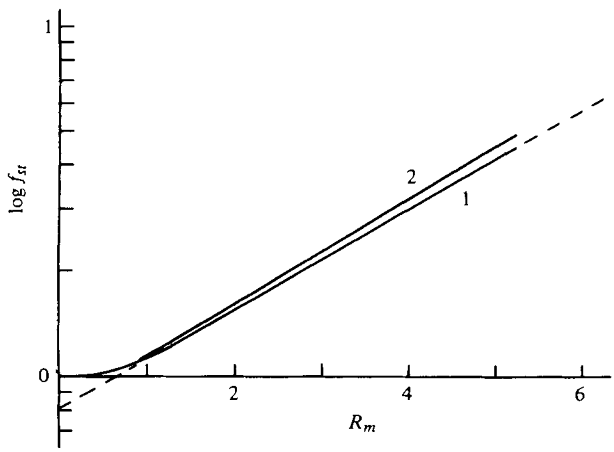

FIGURE 1. Solid lines: f_{st}, the steady-state ratio of magnetic fields below and above the Bénard layer, *vs.* R_m; 1, $h_1 = h_2 - b = 0$ (see I); 2, $h_{1,2} \gg b$ (for an explanation of $h_{1,2}$ see figure 2). Dashed line: approximation $\log f_{st} = -0{\cdot}09 + 0{\cdot}142\ R_m$.

were analysed by Proctor (see Moffatt 1976), Rädler (1976), Drobyshevski (1977), Krause (1978), Vainshtein (1978), and Ruzmaikin & Vainshtein (1978).

The existence of the topological pumping effect was demonstrated in I by a straightforward numerical calculation of the stationary-field distribution in a layer with a given motion structure. H. K. Moffatt, in an appendix to I (see also Moffatt 1978), confirmed the validity of these results for magnetic Reynolds numbers $R_m \ll 1$ using the methods of mean-field electrodynamics and showed the topological pumping to be a third-order effect relative to R_m. Rädler (1976) showed that this effect can also be described in terms of the ordinary α-effect, namely, its antisymmetric part (see Moffatt 1978, p. 150). Besides, Moffatt (1978) pointed out that for description of the strong concentration of the magnetic flux to the lower-layer boundary at $R_m \gg 1$ one might try to apply boundary-layer methods.

Calculation of the steady-state field distribution in a layer yields the magnitude of the relative magnetic field drop $-f_{st}$ produced and maintained by pumping (figure 1, f_{st} is equal to the *stationary* equilibrium ratio of the magnetic field strengths below and above the layer). To derive numerical estimates, one should know also another parameter of the 'pump', viz. its efficiency, i.e. the rate of magnetic field pumping or escape through the layer as a function of the field drop at the given moment.

Parker (1975) attempted to estimate the rate of magnetic field escape through a cellular convection layer. He considered a specific case of a zero field on one side of the layer and came to the conclusion that in this case, topological pumping cannot totally retain the magnetic flux in a half-space at a finite conductivity of the medium. This result is not inconsistent with results to be presented here, since at finite R_m the effect creates and maintains across the layer a finite (rather than infinite) magnetic field ratio.

Having in mind a turbulent medium, Parker concluded that the time characteristic of magnetic flux loss in this case should be determined by the magnitude of the turbulent diffusion coefficient $D_t \approx 0{\cdot}1\ vl$:

$$\tau_R \approx L^2/D_t \qquad (1)$$

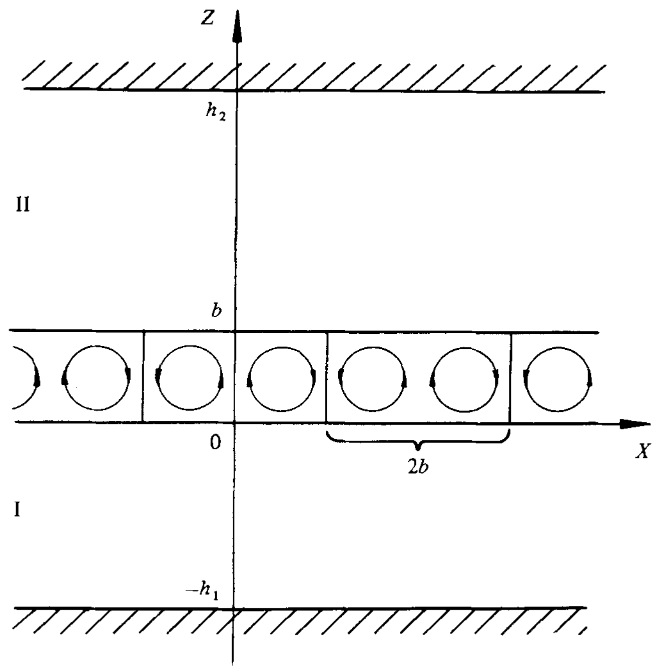

FIGURE 2. Geometry under study: the field B_1 diffuses through the Bénard layer ($0 < Z < b$) from region I into region II, giving rise to the field B_2 there; $Z = -h_1$, $Z = h_2$ are superconducting boundaries.

where v and l are the characteristic turbulence velocity and scale, and L is the size of the body. It is difficult to agree with this conclusion because the topological asymmetry of the velocity field should inevitably lead to asymmetry in the transport properties of the system which is in no way expressed by the estimate (1).

The present work deals with effective transport properties of a cellular convection layer with respect to a magnetic field parallel to the layer. Computational methods and methods of mean field electrodynamics are used to study the integral characteristics of magnetic flux penetration through a convective layer. An symmetry in the transport properties of cellular convection is again demonstrated.

2. Formulation of the problem

To describe the process of magnetic field penetration (or escape) through a plane layer one should, in general, solve a non-stationary problem. Consider the process involved in the simplest geometry of figure 2.

A part of space, $-h_1 \leqslant Z \leqslant h_2$, is bounded by superconducting walls. The layer $0 < Z < b$ contains a conducting medium residing in a state of Bénard convection. Outside this layer is vacuum. The OZ axis is the axis of one of the convective cells. The OX axis lies in the plane of the figure, and OY axis is normal to it.

Just as done earlier (see I), we assume the velocity field in the Bénard layer to be given, and not to be subjected to the back reaction of the magnetic field. This is valid provided the convective velocity is much higher than the Alfvèn velocity (as, for example, on the Sun, outside active regions). Let

$$\mathbf{V} = \pm[-\sin X(1+\tfrac{1}{2}\cos Y)\cos Z, \\ -(1+\tfrac{1}{2}\cos X)\sin Y\cos Z, \ (\cos X+\cos Y+\cos X\cos Y)\sin Z]. \quad (2)$$

Here X, Y, Z are relative to some length scale L. With these variables, $b = \pi$, the central convective cell occupying the space

$$0 \leqslant Z \leqslant \pi, \quad -\pi \leqslant X \leqslant \pi, \quad -\pi \leqslant Y \leqslant \pi.$$

One can consider various situations. One may, for instance, specify the initial homogeneous field $\mathbf{B}_{10}(B_{10X}, 0, 0)$ in region I $(-h_1 < Z < 0)$ and follow the decrease of the field B_1 and its flux $\Phi_1 = B_1 h_1$ and the appearance and increase of the field B_2 and flux $\Phi_2 = B_2(h_2-b)$ in region II $(b < Z < h_2)$. (Here B_1 and B_2-X components of the magnetic field outside the Bénard layer averaged in plane $Z =$ constant. We define below f as $f = B_2/B_1$; but keeping for convenience $f_{st} \geqslant 1$, we have to put in the stationary case $f_{st} = B_2/B_1$ if $B_2 > B_1$, but $f_{st} = B_1/B_2$ if $B_2 < B_1$.)

In regions I and II the magnetic field is defined by equations

$$[\nabla\times\mathbf{B}] = 0, \quad \nabla.\mathbf{B} = 0$$

or introducing the scalar potential ψ:

$$\mathbf{B} = \nabla\psi, \quad \nabla^2\psi = 0. \quad (3)$$

The behaviour of the magnetic field in a convective layer in the velocity field specified is fully described by the following equations

$$\frac{\partial\mathbf{B}}{\partial t} = \nabla^2\mathbf{B} + R_m[\nabla\times(\mathbf{V}\times\mathbf{B})], \quad \nabla.\mathbf{B} = 0. \quad (4)$$

Here $R_m = VL/D$, the quantity $\tau_D = L^2/D$ is chosen as a time scale, while D, assumed to be constant, is the background magnetic diffusivity of medium and is taken as a diffusion coefficient scale. When the convection is laminar, $D = (\mu_0\sigma)^{-1}$, σ is the conductivity, and $\mu_0 = 4\pi\times10^{-7}$ (S.I. units). When the regular cellular motions are superimposed onto a small-scale random turbulence, $D = D_t \approx 0{\cdot}1\,vl$, v and l are velocity and space scales of this turbulence. The magnetic Reynolds number R_m can take on negative values corresponding to a reversal of the velocity field. Because of the length-scale, R_m here is π times that of paper I.

Below, the exact numerical solution of equations (3) and (4) will be carried out. However, to do estimations of the generation rate of magnetic field, of its dissipation, etc., under various conditions there is no need to know all the fine details of the field interaction with the moving medium at every moment. As a rule, it is enough for this purpose to know some mean integral parameters characterizing the process.

So, the magnetic field penetration through the Bénard layer, in the presence of topological pumping (see figure 2), when the averaged field has only one component $\bar{B}_x(Z)$ in the layer, can be described in terms of mean-field electrodynamics (Moffatt 1978) making use of the averaged equation (4)

$$\frac{\partial\bar{B}_x}{\partial t} = \frac{\partial^2\bar{B}_x}{\partial Z^2} - \frac{\partial\mathscr{E}_Y}{\partial Z} \quad (5)$$

where $\mathscr{E}_y = R_m[\overline{\mathbf{V}\times\mathbf{B}'}]_y$ – an effective e.m.f., $\mathbf{B}'$ – the magnetic field perturbation due to the fluid motion. Hence, one can see that the quantity

$$\phi = -\frac{\partial \bar{B}_x}{\partial Z}+\mathscr{E}_y \tag{6}$$

describes a *transport flux* of the averaged field through the convective layer.

If the dependence of $\mathscr{E}_y$ on $\bar{B}_x$ were known, then one would get the solution of the problem formulated above. In the general case (e.g. Rädler 1976),

$$\mathscr{E}_y = \int_0^\pi K(Z,\zeta)\,\bar{B}_x(\zeta)\,d\zeta.$$

But since in our case the space scale of the field $\bar{B}_x$ is the same as the scale of the velocity, i.e. the convective-layer thickness, we cannot express $\mathscr{E}_y(Z)$ in the local form

$$\mathscr{E}_y(Z) = \alpha\bar{B}_x(Z)-\beta\frac{\partial \bar{B}_x}{\partial Z} \tag{7}$$

appropriate for small scale-turbulent fields. (Here the first term to the right describes the α-effect, and the second the enhancement of the diffusion transport. The pumping effect of interest to us arises from the antisymmetric part of the α-effect (Rädler 1976; Moffatt 1978); below we shall designate this part by $-\gamma$.)

However, when the transport process is quasistationary ($h_{1,2} \gg b$), the transport flux ϕ will be practically constant across the layer ($\phi \approx -d\Phi_1/dt \approx d\Phi_2/dt$). If this is the case, then one can attempt to express this flux in terms of an expression which is analogous to (7) but depends on the mean field values at the convective layer boundaries only. So, the main aim of the present work is to look for such a description based on the exact numerical solutions of the full equations (3) and (4).

3. Boundary conditions and method of numerical solution

At $Z = 0$ and $Z = \pi$ the solutions of equations (3) and (4), due to the absence of surface currents, should satisfy the conditions of magnetic field continuity at the interface between two media

$$\mathbf{B}|_- = \mathbf{B}|_+,\quad \left.\frac{\partial B_z}{\partial Z}\right|_- = \left.\frac{\partial B_z}{\partial Z}\right|_+. \tag{8}$$

The boundary conditions for our problem are determined by the existence of superconducting walls:

$$\text{at } Z = -h_1, h_2: \qquad B_z = 0,\quad \frac{\partial B_x}{\partial Z} = 0,\quad \frac{\partial B_y}{\partial Z} = 0. \tag{9}$$

Since the problem is periodic in X and Y, we shall look for a solution in the form of a series

$$\left.\begin{aligned} B_x &= \sum_{i,j=0}^{\infty} B^x_{ij}\cos(iX)\cos(jY),\\ B_y &= \sum_{i,j=0}^{\infty} B^y_{ij}\sin(iX)\sin(jY),\\ B_z &= \sum_{i,j=0}^{\infty} B^z_{ij}\sin(iX)\cos(jY).\end{aligned}\right\} \tag{10}$$

Note that at $i = 0, j = 0$ the only non-zero component is B^x_{00}, and it is this component which determines the average magnetic field over the cell cross-section normal to the OZ axis (i.e. $B^x_{00} = \bar{B}_x$).

Using equations (10), the set (4) can be transformed to

$$\frac{\partial B^x_{ij}}{\partial t} = \frac{\partial^2 B^x_{ij}}{\partial Z^2} - m^2 B^x_{ij} + R_m F^x_{ij} \tag{11a}$$

where

$$F^x_{ij} = (\mathbf{B} \,.\, \nabla V_x - \mathbf{V} \,.\, \nabla B_x)_{ij}, \quad m = (i^2+j^2)^{\frac{1}{2}}. \tag{11b}$$

The equations for B^y_{ij} and B^z_{ij} are of a similar form.

For the non-conducting regions I and II one can find a solution in explicit form

$$\psi = CX + DY + \sum_{i,j=0}^{\infty} (G_{ij} e^{-mZ} + H_{ij} e^{mZ}) \sin(iX) \cos(jY). \tag{12}$$

Using the conditions (9), one readily obtains the following conditions that are valid at the boundaries of the convective layer

$$\left.\begin{aligned} \frac{\partial B^z_{ij}}{\partial Z} &= m\alpha_{ij} B^z_{ij}, \\ B^x_{ij} &= \frac{i}{m} \alpha_{ij} B^z_{ij}, \\ B^y_{ij} &= -\frac{i}{m} \alpha_{ij} B^z_{ij}. \end{aligned}\right\} \tag{13}$$

Here

$$\alpha_{ij} = \begin{cases} \dfrac{1+e^{-2mh_1}}{1-e^{-2mh_1}} & \text{at} \quad Z = 0, \\[2ex] -\dfrac{1+e^{-2m(h_2-\pi)}}{1-e^{-2m(h_2-\pi)}} & \text{at} \quad Z = \pi. \end{cases}$$

This implies that the stationary relative drop f_{st} of magnetic field across the Bénard layer should, generally speaking, depend on the values of $h_{1,2}$.

Taking into account (8), equations (13) provide boundary conditions for the set of transport equations (11) for all field components with $m \neq 0$. To obtain boundary conditions for B^x_{00}, we proceed in the following way. First, we note that in vacuum $B^x_{00} = B_{1,2}$ (functions of t only). Next, we integrate the Maxwell equation

$$\frac{\partial B_x}{\partial t} = -\,[\nabla \times \mathbf{E}]_x$$

over the volume of each non-conducting region (within one cell). Then, because of symmetry of the problem and the presence of superconducting walls, we obtain

$$\frac{\partial B^x_{00}(0)}{\partial t} = \frac{1}{4\pi^2 h_1} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} E_y|_{Z=0}\, dX\, dY,$$

$$\frac{\partial B^x_{00}(\pi)}{\partial t} = -\frac{1}{4\pi^2(h_2-\pi)} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} E_y|_{Z=\pi}\, dX\, dY.$$

Since at boundaries $V_z = 0$,

$$E_y = \left(\frac{\partial B_x}{\partial Z} - \frac{\partial B_z}{\partial X}\right) + R_m V_x B_z$$

we have finally:

$$\left.\begin{aligned} &\text{at } Z = 0+, \quad \frac{\partial B^x_{00}}{\partial t} = \frac{R_m}{h_1}(V_x B_z)_{00} + \frac{1}{h_1}\frac{\partial B^x_{00}}{\partial Z}, \\ &\text{at } Z = \pi-, \quad \frac{\partial B^x_{00}}{\partial t} = -\frac{R_m}{h_2-\pi}(V_x B_z)_{00} - \frac{1}{h_2-\pi}\frac{\partial B^x_{00}}{\partial Z}. \end{aligned}\right\} \quad (14)$$

Comparing (6) with (14), we obtain

$$\left.\begin{aligned} &\text{at } Z = 0, \quad \frac{\partial B^x_{00}}{\partial t} h_1 = \frac{\partial B_1}{\partial t} h_1 = -\phi; \\ &\text{at } Z = \pi, \quad \frac{\partial B^x_{00}}{\partial t}(h_2-\pi) = \frac{\partial B_2}{\partial t}(h_2-\pi) = \phi. \end{aligned}\right\} \quad (15)$$

With our boundary conditions the net magnetic flux of the X component remains constant:

$$\Phi_x = \int_{-\pi}^{\pi}\int_{-h_1}^{h_2} B_x \, dY \, dZ = \Phi_0 = \text{const.} \quad (16)$$

while similar fluxes in each of regions I and II vary.

Equations (11) with boundary conditions (13)–(14) were solved by the finite-difference method using the Crank & Nicolson (1947) technique, with all $i, j > N$ harmonics assumed to be zero. The magnitude of N was chosen such that the higher harmonics be sufficiently small.

Since boundary conditions (13)–(14) are uniform, we added in each iteration of each step in time to B^x_{00} a correction

$$\Delta B^x_{00} = (\Phi_0 - \Phi_x)/(h_2 + h_1),$$

in order to maintain the total flux Φ_0 constant.

4. Magnetic flux behaviour at $|R_m| \ll 1$

Before turning to a discussion of the results of our calculations, we consider some conclusions which may be derived by the methods of mean-field electrodynamics by looking for a solution in a form of a series in powers of small R_m, similar to that used by Moffatt (Appendix to Drobyshevski & Yuferev 1974) to support the existence of the topological pumping phenomenon.

The problem of the field transport can be studied in the steady-state version when $h_{1,2} \to \infty$, so that the ratio $f = B_2/B_1$ specified for the B^x_{00} harmonic at the Bénard layer boundaries remains constant indefinitely. Under these conditions, the continuous transport of field from one region in another takes place.

Performing double integration of (11) for B^x_{00} and taking account of the expressions (15) we obtain

$$\phi|_{Z=0,\pi} = -\frac{B_2 - B_1}{\pi} + \frac{R_m}{8}\int_0^{\pi} [(4B^x_{01} + 4B^x_{10} + 2B^x_{11})\sin Z + 4(B^z_{10} + B^z_{11})\cos Z] \, dZ. \quad (17)$$

To calculate integrals in the preceding expression we again integrate (11 *a*) for corresponding harmonics. Then we have

$$\phi|_{Z=0,\pi} = -\frac{B_2-B_1}{\pi}+\frac{R_m{}^2}{8}\int_0^\pi (2F^x_{01}\sin Z+\tfrac{2}{3}F^x_{11}\sin Z+\tfrac{2}{3}F^z_{11}\cos Z)\,dZ. \tag{18}$$

Introducing here expressions (11 *b*) for F_{ij} we get

$$\phi|_{Z=0,\pi} = -\frac{B_2-B_1}{\pi}+\frac{R_m{}^2}{8}\int_0^\pi [(B^z_{11}+\tfrac{5}{6}B^z_{10})\cos^2 Z + (\tfrac{14}{3}B^x_{00}+\tfrac{5}{2}B^x_{10}+\tfrac{1}{3}B^x_{01}+B^x_{11}-B^y_{11})\sin Z\cos Z]\,dZ \tag{19}$$

+ terms containing harmonics with $i,j > 1$.

Now we expand $\mathbf{B}$ in a series in powers of R_m

$$\mathbf{B} = \sum_{k=0}^{\infty} R_m{}^k\,\mathbf{B}_k \tag{20}$$

where

$$-\nabla^2\mathbf{B}_{k+1} = [\nabla\times[\mathbf{V}\times\mathbf{B}_k]] \tag{21}$$

and $\mathbf{B}_0$ has only one X component

$$B_0 = B^x_{00} = B_1+\frac{B_2-B_1}{\pi}Z. \tag{22}$$

Correspondingly for transport flux ϕ we have

$$\phi = \sum_{k=0}^{\infty} R_m{}^k\,\phi_k. \tag{23}$$

Obviously, $\phi_0 = -(B_2-B_1)/\pi$ and $\phi_1 = 0$. The magnitude of ϕ_2 is found easily by substituting (22) in (19). Hence $\phi_2 = 7\phi_0/48$.

To determine ϕ_3 one has to find $\mathbf{B}_1$ from (21). It is enough to get only particular solution because the general solution of this equation does not give a contribution into expression (19). Then, after simple but tedious transformations we find $\mathbf{B}_1$, and finally

$$\phi_3 = \frac{-5}{72}\frac{B_2+B_1}{\pi}.$$

Thus

$$\phi = -\frac{B_2-B_1}{\pi}\left(1+\frac{7}{48}R_m{}^2+\ldots\right)-\frac{B_2+B_1}{\pi}\left(\frac{5}{72\pi}R_m{}^3+\ldots\right) \tag{24}$$

and in α-effect language (see (6) and (7)), one can write the transport flux at $|R_m| \ll 1$ in form

$$\phi = -\gamma\langle B\rangle-(1+\beta)\left\langle\frac{dB}{dZ}\right\rangle \tag{25}$$

where

$$\langle B\rangle = \tfrac{1}{2}(B_1+B_2) = \tfrac{1}{2}B_1(1+f),$$
$$\langle dB/dZ\rangle = (B_2-B_1)/\pi = B_1(f-1)/\pi,$$
$$1+\beta = 1+\tfrac{7}{48}R_m{}^2+\text{even powers of } R_m \text{ terms},$$
$$\gamma = \tfrac{5}{36}\pi^{-1}R_m{}^3+\text{odd powers of } R_m \text{ terms}.$$

This result immediately implies that the field transport through the Bénard layer depends on the direction of motion ($R_m \gtrless 0$) and the magnitude of the relative field drop f. At $R_m < 0 (0 < f < 1)$, when the velocity at the cell axis is negative, the motion favours field escape and the latter increases monotonically with increasing $|R_m|$. In the opposite case of $R_m > 0 (0 < f < 1)$, the rate of the field escape increases only at small R_m when the second (*even*) harmonic is essential and the convective transport of the field (= 'turbulent' or, more strictly, 'convective' diffusion) prevails over the α-effect action. At $R_m \approx 1{\cdot}4(1-f)/(1+f)$, ϕ reaches a weak maximum after which the topological confinement of the field becomes dominant and the rate of the field escape begins to decrease (for further details see § 5 and figure 5).

These inferences are in qualitative agreement with the results of Weiss (1966), who found that the even harmonic movements expel the field out of medium, and also with the results of Proctor (see Moffatt 1976, 1978), who showed that at $R_m \gg 1$ two-dimensional (even) effects become inessential and the influence of the three-dimensional motions should be most effective here.

Note also, that both β and γ are functions of $R_m (\ll 1)$ only, not f.

5. Results of computations at $|R_m| \gtrsim 1$: The effective values of the pumping rate γ and the convective diffusion β

Numerical calculations permit one to study magnetic field behaviour in a Bénard layer at $|R_m| > 1$ when the presentation of solution in the series form (23)–(24) becomes invalid.

Calculations were performed with $h_{1,2} \gg b = \pi$ to meet the condition of the field distribution quasi-stationarity in the layer. We investigated cases $h_1 = h_2 - \pi = 50$ at different $f = B_2/B_1$; to simulate the field escape into vacuum, where $B_2 = 0 (f = 0)$, a series of runs was done at $h_1 = 10^2$, $h_2 = 10^4$ and $B_2 = 0$ at $t = 0$, the initial stage of process being considered when $f \ll 1$. To make the process reach the quasi-stationary diffusion mode as fast as possible (before the field in region II has attained a noticeable level), the initial field distribution inside the layer was assumed to be linear, $B = 1 - Z/\pi$.

We are interested here mainly in integral characteristics of the process and therefore in contrast to paper I, we will not go into details of the field distribution inside the Bénard layer. In the initial stages, motions inside the layer rebuild strongly the field configuration pushing some part of the magnetic flux out of the layer (dominantly in the direction of pumping). But by time $t = 3$, practically quasi-stationary conditions are reached. Here the field distribution in the layer depends strongly on the direction of motion in the cells. For $R_m < 0$ the averaged magnetic field distribution in the layer is closer to the linear one and the gradients $\nabla_z \bar{B}_x$ near the boundaries are larger than those for $R_m > 0$ (figure 3). Since near the boundaries the field transport occurs mainly through Ohmic dissipation, it follows, in accordance with topological considerations that field penetration through the layer will in the case $R_m < 0$ occur, on the whole, faster, than for $R_m > 0$.

The results of the exact numerical solution allow us to define parameters characterizing integral properties of the Bénard layer, namely the additional 'convective' diffusion coefficient β and the pumping-rate coefficient γ. For this purpose we will make use of the equation of type (25) in the form

$$\phi/B_1 = -\tfrac{1}{2}\gamma(f+1) - (1+\beta)(f-1)/\pi. \tag{26}$$

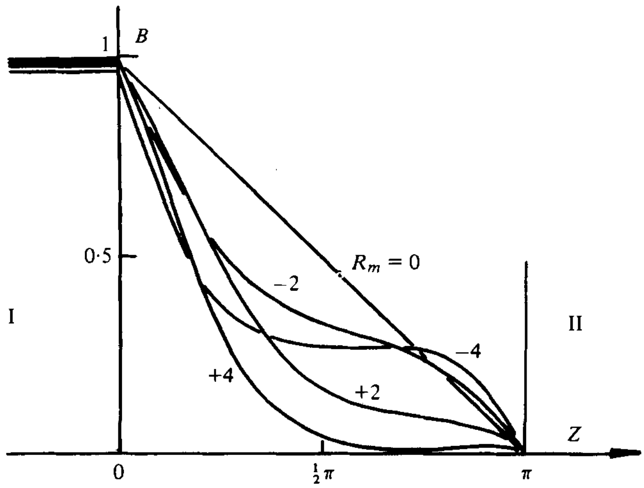

FIGURE 3. Steady-state distribution of the averaged field B_x in the Bénard layer at $f = B_2/B_1 \approx 0$ ($t = 5$, $h_1 = 100$, $h_2 = 10000$).

Since in this case $|R_m| \gtrsim 1$ the equation (26) is obtained from (6) and (7) by means of simple replacement of the local values $\bar{B}_x$ and $\nabla_z \bar{B}_x$ by values expressed via the mean field values on the layer boundaries, one cannot in general exclude dependence of β and γ on f (although the first terms of expansions (23)–(24) at $|R_m| \ll 1$ do not contain such dependence).

On the other hand, it is natural to require that when R_m changes sign, other things being equal, β keeps its value and remains positive (this corresponds to β expansion in series of even powers), while γ, preserving its absolute value, changes its sign (the expansion in odd power series). Then at f fixed one obtains from (26)

$$\left.\begin{aligned}(\phi/B_1)_{Rm>0} + (\phi/B_1)_{Rm<0} &= -\frac{2}{\pi}(f-1)(1+\beta),\\ (\phi/B_1)_{Rm>0} - (\phi/B_1)_{Rm<0} &= -(f+1)\gamma.\end{aligned}\right\} \quad (27)$$

The treatment of the exact calculation results making use of equations (27) permits evaluation of β and γ. These results are presented in figure 4. Note that in the limits of our R_m range ($-5 \leqslant R_m \leqslant 5$) and computation accuracy ($\sim 1\%$) β and γ do not depend on f. Then as soon as $\phi \equiv 0$ at $f = 1/f_{st}$ ($R_m > 0$), it follows from (26) that

$$\gamma/(1+\beta) = (2/\pi)(f_{st}-1)/(f_{st}+1).$$

(Note that at $h_{1,2} \gg b$, f_{st} exceeds slightly (up to 8% at $R_m = 5$) f_{st} found for

$$h_1 = h_2 - \pi = 0$$

(cf. I); see figure 1.) In the limiting case $R_m \to \infty$, $f_{st} \to \infty$, $\gamma/(1+\beta) = 2/\pi$, and $\phi \to 0$, i.e. the topological pumping effect suppresses fully the field escape into vacuum. Obviously, at the external field presence ($f > 0$, $R_m > 0$) the blocking of the field escape from the region I takes place at $f = 1/f_{st}$. At $f = 1$ (equal fields in regions I and II) the diffusion stops to work and the field transport goes on by virtue of the topological effect only.

The calculations show that equation (26) with the coefficients β and γ found above

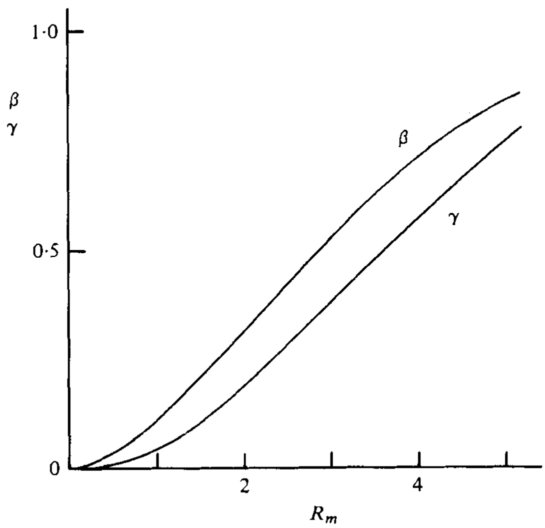

FIGURE 4. The dependence of the 'convective' diffusion coefficient β and the pumping rate coefficient γ (see equations (25) and (26)) on R_m.

describe the field transport process at $f < 0$ as well (the fields of opposite directions on different sides of the convective layer).

It is of interest to clear up at what values of parameters (R_m and $f = f_{cr}$) the topological pumping compensates fully the effect of the diffusion enhancement caused by convective motions and, moreover, begins to suppress the magnetic field escape due to the initial background diffusion, i.e. when is the condition stated below satisfied?

$$\phi/B_1 = -\tfrac{1}{2}\gamma(f+1)-(1+\beta)(f-1)/\pi \leqslant -(f-1)/\pi. \tag{28}$$

(The transport flux of the magnetic field in the presence of convection is equal to or smaller than the flux in absence of the convection; the equality is achieved at $f = f_{cr}$.)

The results of the equation (28) solution which takes into account dependencies $\gamma(R_m)$ and $\beta(R_m)$ are presented in figure 5. One can see the topological pumping begins to suppress the field escape into the free vacuum ($f_{cr} = 0$) at $R_m \approx 2{\cdot}2$.

Since at the quasistationary regime the transport flux ϕ is practically constant across the layer, equation (15) can be integrated making use of expression (26). Then

$$B_1 = B_{00}^x(0) = C_1 + C_2 e^{-at}$$

where

$$a = -\frac{\gamma}{2}\left(\frac{1}{h_1}-\frac{1}{h_2-\pi}\right)+\frac{1+\beta}{\pi}\left(\frac{1}{h_1}+\frac{1}{h_2-\pi}\right)$$

and one can see that only at $h_1 = h_2 - \pi$ the time constant of the mean field transport does not depend on α-effect (here $a = 2(1+\beta)/\pi h_1$). On the other side, if $h_2 \to \infty$ (say, the field escapes from the celestial body into free space), then

$$a = \frac{1}{h_1}\left(\frac{1+\beta}{\pi}-\frac{\gamma}{2}\right)$$

and the field escape rate into vacuum depends on the direction of the movements in the convective cells (i.e., on the sign of γ).

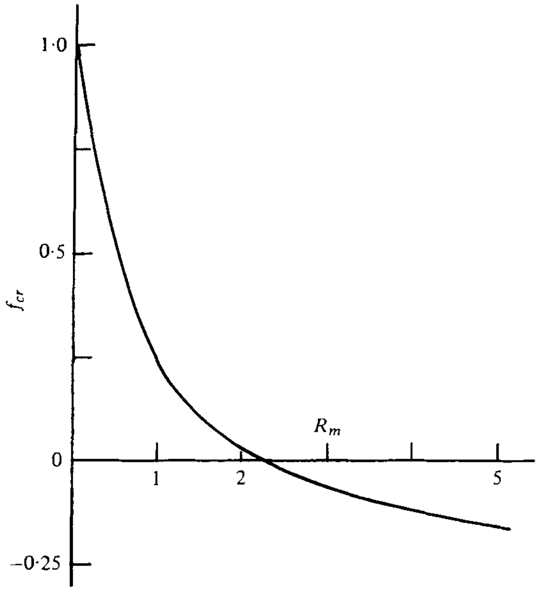

FIGURE 5. f_{cr}, the value of $f = B_2/B_1$ corresponding to the exact compensation of the 'convective' diffusion by the pumping effect. In the region over the curve, the topological pumping makes the magnetic field escape through the convective layer to be smaller than the escape through the layer having no convection.

6. Conclusion: On the transport under conditions of turbulent convection

Calculations of magnetic field penetration through a layer with a cell structure of motion show the topological properties of motion to affect strongly the process. The effective transport properties of the Bénard layer turn out to be asymmetrical and dependent on magnetic Reynolds number and its sign.

From the equation (26) taking into account relations $\gamma(R_m)$ and $\beta(R_m)$ (figure 4) it follows that the flux escape rate through the layer grows monotonically with $|R_m|$ and is maximal when the motion in the cells occurs in the direction favouring field escape ($R_m < 0$) and the counter-field is absent at all. Here both the convective diffusion (β) enhancement and the topological pumping (γ) are acting in one direction.

On the contrary, at $R_m > 0$ the pumping lays obstacles to the field escape and at $R_m > 2{\cdot}2$ (and $f = 0$) the topological pumping action compensates fully the diffusion carrying-out due to convective movements, and moreover, it compensates in a considerable degree even the initial background diffusion input into the magnetic field transport. This compensation grows monotonically with increasing R_m and, as a result, the magnetic field escape decreases strongly when compared with the field escape through a layer having no cell motions.

These results are in contradiction with the conclusion of Parker (1975) that motion topology does not affect magnetic field escape rate from a turbulent gas body into free space and the escape rate is governed only by the diffusion coefficient $D_t \approx 0{\cdot}1\, vl$, where v and l are the velocity and the size of the dominant eddies, no matter what is their topology. The proved dependence of the magnetic field *transport* flux ϕ ($= d\Phi/dt$) on the magnitude and sign of R_m caused by the topological properties of cellular motion illustrates unambiguously the fundamental difference in transport properties between convection and turbulence.

It is instructive also to elucidate one point related to the value of the background diffusion coefficient in the presence of the topological effect in turbulized media. Admitting the possibility of topological action of the turbulized medium on magnetic field, which case was studied by Parker, one has to assume the existence of cell structure in the presence of turbulence. In the limiting case these should be quasistationary structures. (Simple physical considerations show that in large scale thermal convection such quasistationarity can possibly occur (Drobyshevski 1971).) Then the background magnetic diffusivity $D = 0{\cdot}1\, vl$ will be determined not by the largest scale of more or less stable structures but rather by the small scale of random turbulence. The latter will be several times smaller than the former, at a conservative estimate, $l \lesssim \frac{1}{3}/b$, $v \lesssim \frac{1}{3}V$. Hence $D \lesssim 0{\cdot}01\ Vb$ which corresponds to the effective value $R_m \gtrsim 30$ (we may recall that $b = \pi$ at $L = 1$). It would be too risky to extrapolate our results (available up to $R_m \approx 5$, see figure 1) so far; but if we do we find that at $R_m = 30$ the steady-state field drop could reach $f_{st} = 10^4$(!) (the field will stop escaping altogether if the ambient field is 10^{-4} of the field in the body). Thus the major conclusion on the difference in transport of magnetic field by cell convection (the background diffusion coefficient is here $D \lesssim 0{\cdot}01\ Vb$) and random turbulence ($D_t \approx 0{\cdot}1\ Vb$) finds additional support.

It appears to be obvious, although it requires further study, that in the intermediate case of nonstationary turbulent convection which has the necessary topological structure (granulation and supergranulation on the Sun, ascension of isolated thermals, etc.) magnetic field transport should likewise be different, although to a lesser degree, from that by isotropic turbulence.

The authors are grateful to Professor H. K. Moffatt and Dr M. R. E. Proctor who suggested that the results should be expressed in the α-effect language, making the presentation more comprehensible and compact.

REFERENCES

Crank, J. & Nicolson, P. 1947 A practical method for numerical evaluation of solutions of partial differential equations of the heat-conduction type. *Proc. Camb. Phil. Soc.* **43**, 50–67.

Drobyshevski, E. M. 1971 On the structure of motions in convective envelope of the Sun. *Soln. Dannye* (*Solar Data*) **6**, 84–88.

Drobyshevski, E. M. 1977 Magnetic field transfer by two-dimensional convection and solar 'semi-dynamo'. *Astrophys. & Space Sci.* **46**, 41–49.

Drobyshevski, E. M. & Yuferev, V. S. 1974 Topological pumping of magnetic flux by three-dimensional convection. *J. Fluid Mech.* **65**, 33–44.

Krause, F. 1978 The pumping effect in mean-field magnetohydrodynamics, *Publ. Astron. Inst. Czechoslov.* **54**, 41–42.

Moffatt, H. K. 1976 Generation of magnetic fields by fluid motion. *Adv. Appl. Mech.* **16**, 119–181.

Moffatt, H. K. 1978 *Magnetic Field Generation in Electrically Conducting Fluids.* Cambridge University Press.

Parker, E. N. 1975 The escape of magnetic flux from a turbulent body of gas. *Astrophys. J.* **202**, 523–527.

Rädler, H.-K. 1976 Mean-field magnetohydrodynamics as a basis of solar dynamo theory. In *Basic Mechanisms of Solar Activity* (ed. V. Bumba & J. Kleczek), pp. 323–344. Dordrecht, Holland: Reidel.

Ruzmaikin, A. A. & Vainshtein, S. I. 1978 Magnetic field transport in the solar convective zone. *Astrophys. & Space Sci.* **57**, 195–202.

Vainshtein, S. I. 1978 MHD effects in turbulized medium with nonhomogeneous density. *Magnitnaya Gidrodinamika* **1**, 45–50.

Weiss, N. O. 1966 The expulsion of magnetic flux by eddies. *Proc. Roy. Soc.* A **293**, 310–328.